Cholesky Decomposition and MVN

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1 Problem Setup

Let $\mathbf{x}$ be a d-dimensional random variable having a Gaussian distribution with zero mean and identity covariance matrix, and suppose that the positive definite symmetric matrix $\mathbf{\Sigma}$ has the following Cholesky decomposition:

$$\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^\top$$

where $\mathbf{L}$ is a lower-triangle matrix. Show that the new variable $\mathbf{y} = \boldsymbol{\mu} + \mathbf{L}\mathbf{x}$ has a Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$.

1.1 Solution

We know that $\mathbb{E}(\mathbf{x}) = \mathbf{0}$, therefore,

$$\mathbb{E}(\mathbf{y}) = \mathbb{E}(\boldsymbol{\mu} + \mathbf{L}\mathbf{x}) = \boldsymbol{\mu}$$

Also $\mathbf{x}$ has identity covariance, which means:

$$Cov(\mathbf{x}) = \mathbb{E}(\mathbf{x}\mathbf{x}^\top) - \mathbb{E}(\mathbf{x})\mathbb{E}(\mathbf{x}^\top) = \mathbb{E}(\mathbf{x}\mathbf{x}^\top) = \mathbf{I}_{d\times d}$$

Therefore,

$$\begin{aligned}
Cov(\mathbf{y}) &= \mathbb{E}(\mathbf{y}\mathbf{y}^\top) - \mathbb{E}(\mathbf{y})\mathbb{E}(\mathbf{y}^\top) \\
&= \mathbb{E}\left[(\boldsymbol{\mu} + \mathbf{L}\mathbf{x})(\boldsymbol{\mu} + \mathbf{L}\mathbf{x})^\top\right] - \boldsymbol{\mu}\boldsymbol{\mu}^\top \\
&= \mathbb{E}\left[\boldsymbol{\mu}\boldsymbol{\mu}^\top + \boldsymbol{\mu}\mathbf{x}^\top\mathbf{L}^\top + \mathbf{L}\mathbf{x}\boldsymbol{\mu}^\top + \mathbf{L}\mathbf{x}\mathbf{x}^\top\mathbf{L}^\top\right] - \boldsymbol{\mu}\boldsymbol{\mu}^\top \\
&= \boldsymbol{\mu}\boldsymbol{\mu}^\top + \mathbf{0}_{d\times d} + \mathbf{0}_{d\times d} + \mathbf{L}\mathbb{E}(\mathbf{x}\mathbf{x}^\top)\mathbf{L}^\top - \boldsymbol{\mu}\boldsymbol{\mu}^\top \\
&= \mathbf{L}\mathbf{L}^\top \\
&= \mathbf{\Sigma}
\end{aligned}$$

Q.E.D.

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