Proof of the Dual Equilateral Triangle Problem - Simplicity is Hidden Complexity

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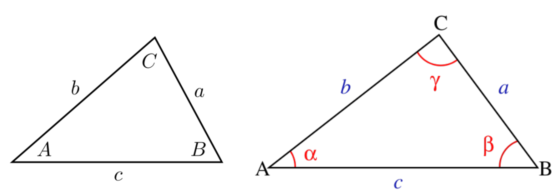
1 Law of Sines Review

The law of sines [1] relates the lengths of the sides of an arbitrary triangle to the sines of its angles:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{1}$$

where a, b, c are the lengths of the sides of a triangle and A, B, C are the opposite angles as shown in Figure 1(a). R is the radius of the circumcircle of ΔABC.



(a) (b)

Figure 1: (a) Illustration of the law of sines [1], and (b) illustration of the law of cosines [2].

Corollary 1.1. *Larger angle corresponds to longer opposite side.*

Proof. If $\frac{\pi}{2} \geq C > B \geq A > 0$, then according to the law of sines, we can have $c > b \geq a$.

Else if $\pi > C > \frac{\pi}{2} > B \geq A > 0$. Let $C = \frac{\pi}{2} + \tau$, where $\frac{\pi}{2} > \tau > 0$.

Since $A + B + C = \pi$, therefore $B = \pi - A - C = \pi - (\frac{\pi}{2} + \tau) - C = \frac{\pi}{2} - \tau - C < \frac{\pi}{2} - \tau$

$\Rightarrow \sin C \geq \sin B > \sin A \Rightarrow c > b \geq a$.

□

Lemma 1.2. *In ΔABC and $\Delta A'B'C'$, if $\frac{\sin A}{\sin B} = \frac{\sin A'}{\sin B'} = k$ where $k > 0$ is a constant and $0 < \max(A, B, A', B') \leq \frac{\pi}{2}$, then $A' \gtreqless A \Leftrightarrow B' \gtreqless B$.*

Proof. When $0 < \max(A, B, A', B') \leq \frac{\pi}{2}$, $A' > A \Rightarrow \sin A' > \sin A \Rightarrow \sin B' > \sin B \Rightarrow B' > B$. Same for the remaining cases. □

2 Law of Cosines Review

The law of cosines [2] relates the lengths of the sides of a plane triangle to the cosine of one of its angles:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{2}$$

where γ is the angle contained between sides of length a and b and opposite the side of length c as shown in Figure 1(b).

3 Problem

As shown in Figure 2, ΔDEF is an equilateral triangle, and $\overline{BD} = \overline{CE} = \overline{AF}$.

Prove: ΔABC is also an equilateral triangle.

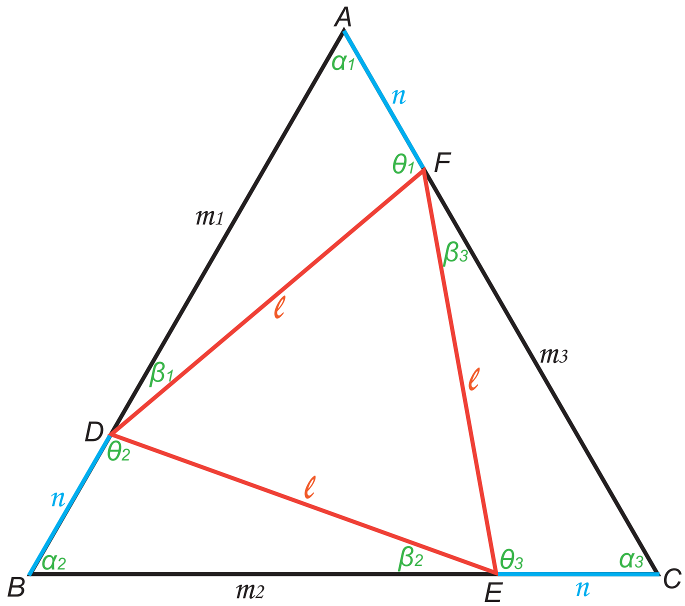


Figure 2: Illustration of the problem.

For simplicity, let $\overline{DE} = \overline{EF} = \overline{FD} = l$, $\overline{BD} = \overline{CE} = \overline{AF} = n$, and $\overline{AD} = m_1$, $\overline{BE} = m_2$, $\overline{CF} = m_3$. Angles are labeled accordingly as shown in Figure 2. Also, without loss of generality, let $n \leq l$.

Proof. Proof by contradiction: suppose $\alpha_1 > \alpha_2 \geq \alpha_3$.

Based on Corollary 1.1 $\Rightarrow \overline{BC} > \overline{AC} \geq \overline{AB}$.

Since $\overline{BD} = \overline{CE} = \overline{AF} = n \Rightarrow m_2 > m_3 \geq m_1$.

In ΔADF, the law of cosines $\Rightarrow m_1^2 = n^2 + l^2 - 2nl\cos\theta_1 \Rightarrow \cos\theta_1 = \frac{n^2+l^2-m_1^2}{2nl}$.

Similarly in ΔBDE and ΔCEF, we can have $\cos\theta_2 = \frac{n^2+l^2-m_2^2}{2nl}$ as well as $\cos\theta_3 = \frac{n^2+l^2-m_3^2}{2nl}$.

$\Rightarrow \theta_2 > \theta_3 \geq \theta_1$.

Again, since $\alpha_1 > \alpha_2 \geq \alpha_3$, we have $\alpha_1 > \frac{\pi}{3}$, $\alpha_3 \leq \frac{\pi}{3}$, and $\alpha_2 < \frac{\pi}{2}$.

Now, let us focus on ΔBDE and ΔCEF, we now have $\max(\alpha_2, \beta_2) < \frac{\pi}{2}$, and $\max(\alpha_3, \beta_3) < \frac{\pi}{3} < \frac{\pi}{2}$. According to Lemma 1.2, $\theta_2 > \theta_3 \Rightarrow \alpha_2 + \beta_2 < \alpha_3 + \beta_3 \Rightarrow \underline{\alpha_2 < \alpha_3}$ and $\beta_2 < \beta_3$.

This contradicts the assumption.

Therefore, $\alpha_3 = \alpha_2 = \alpha_1 \Rightarrow \Delta ABC$ is an equilateral triangle. □

References

[1] Law of Sines, http://en.wikipedia.org/wiki/Law_of_sines

[2] Law of Cosines, http://en.wikipedia.org/wiki/Law_of_cosines